DAY TWENTY SEVEN

Parabola

Learning & Revision for the Day

- Conic Section
- Concept of Parabola
- Line and a Parabola
- Equation of Tangent
- Equation of Normal
- Equation of a Pair of Tangents
- Equations of Chord of Contact
- Director Circle
- Conormal Points
- Diameter

Conic Section

A conic is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is the **focus** S and the fixed line is the **directrix,** l.

The constant ratio is the **eccentricity** denoted by e.

- If $0 < e < 1$, then conic is an ellipse.
- If $e = 1$, then conic is a parabola.
- If $e > 1$, then conic is a hyperbola.

General Equation of Conic Section

A second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

***Case* I** When the focus lies on the directrix

(i) Pair of straight lines, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If $e > 1$, then the lines will be real and distinct intersecting at fixed point.

(iii) If $e = 1$, then the lines will coincident passing through a fixed point.

(iv) If $e < 1$, then the lines will be imaginary.

***Case* II** When the focus does not lie on the directrix

(i) Circle : $a = b$, $h = 0$, $e = 0$ and $\Delta \neq 0$

(ii) Parabola : $h^2 = ab$, $\Delta \neq 0$, $e = 1$

(iii) Ellipse : $h^2 < ab$, $\Delta \neq 0$, $0 < e < 1$

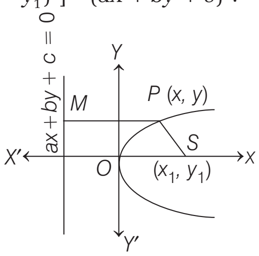
(iv) Hyperbola : $h^2 > ab$, $0\ \Delta \neq 0$, $e > 1$

(v) Rectangular hyperbola : $a + b = 0$, $\Delta \neq 0$, $e > 1$, $h^2 > ab$

Concept of Parabola

Parabola is the locus of a point which moves in a plane such that its distance from a fixed point (focus, S) is equal to its distance from a fixed straight line (directrix, L).

Let $S \equiv (x_1, y_1)$ and $L \equiv ax + by + c = 0$.
Then, equation of parabola is
$(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$.

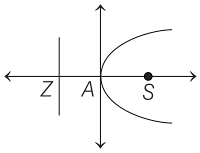
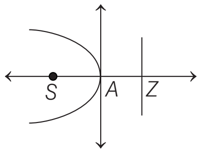
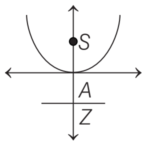
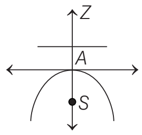


If S lies on L, parabola reduces to a straight line through S and perpendicular to L.

Definitions Related to Parabola

1. **Vertex** The intersection point of parabola and axis.
2. **Centre** The point which bisects every chord of the conic passing through it.
3. **Focal chord** Any chord passing through the focus.
4. **Double ordinate** A chord perpendicular to the axis of a conic.
5. **Latusrectum** A double ordinate passing through the focus of the parabola.
6. **Focal distance** The distance of a point $P(x, y)$ from the focus S is called the focal distance of the point P.

Some related terms of parabolas (in standard form)

S. No.	Related Terms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
					
1.	Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
2.	Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
3.	Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
4.	Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
5.	Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
6.	Extremities of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
7.	Length of latusrectum	$4a$	$4a$	$4a$	$4a$
8.	Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
9.	Parametric equation	$\begin{cases} x = at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = -at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}$	$\begin{cases} x = 2at \\ y = -at^2 \end{cases}$
10.	Focal distance of any point $P(h, k)$ on the parabola	$h + a$	$h - a$	$k + a$	$k - a$
11.	Equation of latusrectum	$x = a$	$x + a = 0$	$y = a$	$y + a = 0$

Results on Parabola $y^2 = 4ax$

(i) Length of latusrectum $= 2$ (Harmonic mean of focal segment)

(ii) If y_1, y_2 and y_3 are the ordinates of the vertices of triangle inscribed in the parabola $y^2 = 4ax$,
then its area $= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

(iii) For the ends of latusrectum of the parabola $y^2 = 4ax$, the values of the parameter are ± 1.

Position of a Point

A point (h, k) with respect to the parabola S lies inside, on or outside the parabola, if $S_1 < 0, S_1 = 0$ or $S_1 > 0$.

Line and a Parabola

(i) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according to $c > \frac{a}{m}$, $c = \frac{a}{m}$ or $c < \frac{a}{m}$ respectively.

(ii) Length of the chord intercepted by the parabola on the line $y = mx + c$ is $= \frac{4\sqrt{a(1+m^2)(a-mc)}}{m^2}$

(iii) Length of the focal chord making an angle α with the X-axis is $4a\operatorname{cosec}^2\alpha$.

(iv) If t_1 and t_2 are the end points of a focal chord of the parabola $y^2 = 4ax$, then $t_1t_2 = -1$

Equation of Tangent

A line which intersects the parabola at only one point is called the tangent to the parabola.

Equation of tangent to parabola in different cases are given below;

- In point (x_1, y_1) form, $yy_1 = 2a(x + x_1)$
- In slope (m) form, $y = mx + \frac{a}{m}$
- In parametric (t) form, $ty = x + at^2$
- The line $y = mx + c$ touches a parabola iff $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Results on Tangent

(i) Points of intersection of tangents at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{at_1t_2, a(t_1 + t_2)\}$ (where, R is GM of x-coordinates of P, Q and AM of y-coordinates of P, Q).

(ii) Angle θ between tangents at two points $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$.

(iii) Locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

(iv) If the tangents at the points P and Q on a parabola meet T, then ST is the GM between SP and SQ i.e. $ST^2 = SP \cdot SQ$

(v) If the tangent and normal at any point P of the parabola intersect the axis at T and G, then $ST = SG = SP$, where S is the focus.

(vi) Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.

(vii) The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

(viii) The length of the subtangent at any point on a parabola is equal to twice the abscissae of the point.

(ix) Two tangents can be drawn from a point to a parabola. Two tangents are real and distinct or coincident or imaginary according as given point lies outside, on or inside the parabola.

Equation of Normal

A line which is perpendicular to the tangent of the parabola is called the normal to the parabola.

Equation of normal to parabola in different cases are given below;

- In point (x_1, y_1) form, $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$.
- In slope m form, $y = mx - 2am - am^3$.
- In parametric t form, $y + tx = 2at + at^3$.

NOTE Point of intersection of normals of t_1 and t_2 are $[a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2)]$.

Results on Normal

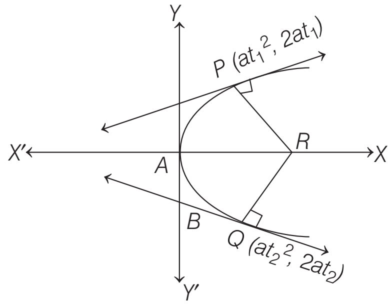
(i) If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersects at a third point R on the curve, then the product of the ordinates of P and Q is $8a^2$.

(ii) Normal at the ends of latusrectum of the parabola $y^2 = 4ax$ meet at right angles on the axis of the parabola.

(iii) Tangents and normals at the extremities of the latusrectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.

(iv) The normal at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.

(v) The normal drawn at a point $P(at_1^2, 2at_1)$ to the parabola $y^2 = 4ax$ meets again the parabola at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.



(vi) The normal chord of a parabola at a point whose ordinate is equal to the abscissae, subtends a right angle at the focus.

(vii) Three normals can be drawn from a point to a parabola.

Equation of a Pair of Tangents

The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the parabola is $SS_1 = T^2$.

where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and

$T = yy_1 - 2a(x + x_1)$

Equations of Chord of Contact

1. The equation of chord of contact is
$$yy_1 - 2a(x + x_1) = 0$$
2. The equation of chord of parabola, whose mid-point (x_1, y_1) is $T = S_1$, i.e. $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
3. Length of the chord of contact is
$$l = \frac{\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}}{a}.$$
4. Area of the ΔPAB formed by the pair of tangents and their chord of contact is
$$A = \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}.$$

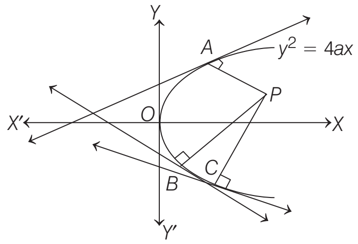
NOTE
- Equation of the chord joining points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ is $(t_1 + t_2)y = 2x + 2at_1t_2$.
- For PQ to be focal chord, $t_1t_2 = -1$.
- Length of the focal chord having t_1, t_2 as end points is $a(t_2 - t_1)^2$.

Director Circle

The locus of the point of intersection of perpendicular tangents to a conic is known as director circle. The director circle of a parabola is its directrix.

Conormal Points

The points on the parabola at which the normals pass through a common point are called conormal points. The conormal points are called the feet of the normals.



Points A, B and C are called conormal points with respect to point P.

1. The algebraic sum of the slopes of the normals at conormals point is 0.
2. The sum of the ordinates of the conormal points is 0.
3. The centroid of the triangle formed by the conormal points on a parabola lies on its axis.

Diameter

Diameter is the locus of mid-points of a system of parallel chords of parabola.

1. The tangent at the extremities of a focal chord intersect at right angles on the directrix and hence a circle on any focal chord as diameter touches the directrix.
2. A circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.
3. The diameter bisecting chords of slope m to the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Equation of the parabola whose vertex is $(-1, -2)$, axis is vertical and which passes through the point $(3, 6)$ is

(a) $x^2 + 4x + 28y - 136 = 0$
(b) $x^2 + 2x - 2y - 3 = 0$
(c) $y^2 + 4y - 16x - 12 = 0$
(d) None of the above

2 A focal chord of the parabola $y^2 = 8x$ in inclined to X-axis at an angle $\tan^{-1} 3$. Then its length is equal to

(a) 80/3 (b) 80/9 (c) 40/3 (d) 40/9

3 Latus rectum of the parabola whose axis is parallel to the Y-axis and which passes through the points (0, 4), (1, 9), and $(-2, 6)$ is equal to

(a) 1/2 (b) 1 (c) 2 (d) None of these

4 If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the value of k is

(a) 1/8 (b) 8 (c) 4 (d) 1/4

5 The locus of trisection point of any double ordinate of the parabola $y^2 = 4ax$ is

(a) $y^2 = 9ax$ (b) $y^2 = ax$
(c) $9y^2 = 4ax$ (d) None of these

6 Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio $1:3$, then the locus of P is

→ **JEE Mains 2015**

(a) $x^2 = y$ (b) $y^2 = x$
(c) $y^2 = 2x$ (d) $x^2 = 2y$

7 At any points P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q the locus of the points R which divides QP externally in the ratio $\frac{1}{2}:1$, is

(a) $(x+1)(1-y)^2 + 4 = 0$ (b) $x + 1 = 0$
(c) $(1-y)^2 - 4 = 0$ (d) None of these

8 The line $x - b + \lambda y = 0$ cuts the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$. If $b \in [2a, 4a]$ and $\lambda \in R$, then $t_1 t_2$ belongs to

(a) $[-4, -2]$ (b) $[4, -3]$
(c) $[-3, -2]$ (d) None of these

9 The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is

(a) $\left(\frac{-16}{5}, \frac{27}{10}\right)$ (b) $\left(\frac{-16}{7}, \frac{53}{10}\right)$
(c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$ (d) None of these

10 Equation of common tangents to parabolas $y = x^2$ and $y = -x^2 + 4x - 4$ is/are

(a) $y = 4(x-1)$; $y = 0$ (b) $y = 0$, $y = -4(x-1)$
(c) $y = 0$, $y = -10(x+5)$ (d) None of these

11 Angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$ is

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

12 If the lines $y - b = m_1(x + a)$ and $y - b = m_2(x + a)$ are the tangents of the parabola $y^2 = 4ax$, then

(a) $m_1 + m_2 = 0$ (b) $m_1 m_2 = 1$
(c) $m_1 m_2 = -1$ (d) $m_1 + m_2 = 1$

13 Set of values of h for which the number of distinct common normals of $(x-2)^2 = 4(y-3)$ and $x^2 + y^2 - 2x - hy - c = 0$ where, $(c > 0)$ is 3, is

(a) $(2, \infty)$ (b) $(4, \infty)$ (c) $(2, 4)$ (d) $(10, \infty)$

14 Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is

→ **JEE Mains 2018**

(a) $\frac{1}{2}$ (b) 2 (c) 3 (d) $\frac{4}{3}$

15 P is a point on the parabola $y^2 = 4x$ and Q is a point on the line $2x + y + 4 = 0$. If the line $x - y + 1 = 0$ is the perpendicular bisector of PQ, then the coordinates of P is

(a) (8, 9), (10, 11) (b) (1, −2), (9, −6)
(c) (7, 8), (9, 8) (d) None of these

16 The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is

(a) $xy = 105/64$ (b) $xy = 3/4$
(c) $xy = 35/16$ (d) $xy = 64/105$

17 The parabola $y^2 = \lambda x$ and $25[(x-3)^2 + (y+2)^2] = (3x - 4y - 2)^2$ are equal, if λ is equal to

(a) 1 (b) 2 (c) 3 (d) 6

18 A line is drawn from $A(-2, 0)$ to intersect the curve $y^2 = 4x$ in P and Q in the first quadrant such the $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$, then slope of the line is always

(a) $< \sqrt{3}$ (b) $> \sqrt{3}$
(c) $\geq \sqrt{3}$ (d) None of these

19 Vertex A of a parabola $y^2 = 4ax$ is joined to any point P on it and line PQ is drawn at right angle to AP to meet the axis at Q. Then, the projection of PQ on the axis is always equal to

(a) $3a$ (b) $2a$
(c) $\sqrt{3}a$ (d) $4a$

20. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is → **JEE Mains 2014**

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{8}$ (d) $\frac{2}{3}$

21 If the normals at the end points of variable chord PQ of the parabola $y^2 - 4y - 2x = 0$ are perpendicular, then the tangents at P and Q will intersect on the line

(a) $x + y = 3$ (b) $3x - 7 = 0$ (c) $y + 3 = 0$ (d) $2x + 5 = 0$

22 Find the length of the normal drawn from the point on the axis of the parabola $y^2 = 8x$ whose distance from the focus is 8.

(a) 10 (b) 8 (c) 9 (d) None of these

23 If $x + y = k$ is a normal to the parabola $y^2 = 12x$, p is the length of the perpendicular from the focus of the parabola on this normal, then $3k^3 + 2p^2$ is equal to

(a) 2223 (b) 2224
(c) 2222 (d) None of these

24 If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b + 3c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b - 3c)^2 = 0$ (d) $d^2 + (3b - 2c)^2 = 0$

25 Slopes of the normals to the parabola $y^2 = 4ax$ intersecting at a point on the axis of the parabola at a distance $4a$ from its vertex are in

(a) HP (b) GP
(c) AP (d) None of these

26 The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$, both drawn at the same end of the latusrectum and the axis of the parabola is

(a) $2\sqrt{2}\,a^2$ (b) $2a^2$
(c) $4a^2$ (d) None of these

27 If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, then mid-point of QR is

(a) (2, 4) (b) (4, 2)
(c) (7, 9) (d) None of these

28 The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the X-axis is

(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x + 3)$
(c) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x + 1)$

29 If tangents drawn from point P to the parabola $y^2 = 4x$ are inclined to X-axis at angles θ_1 and θ_2 such that $\cot\theta_1 + \cot\theta_2 = 2$, then locus of the point P is

(a) $y = 2$ (b) $y = 8$ (c) $y = 1$ (d) None of these

30 Tangents to the parabola $y^2 = 4x$ are drawn from the point (1, 3). The length of chord of contact is

(a) 5 (b) 13 (c) $\sqrt{65}$ (d) None of these

Directions (Q. Nos. 31-35) *Each of these questions contains two statements : Statement* I *(Assertion) and Statement* II *(Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (a), (b), (c) *and* (d) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

31 **Statement I** The perpendicular bisector of the line segment joining the points $(-a, 2at)$ and $(a, 0)$ is tangent to the parabola $y^2 = -4ax$, where $t \in R$.

Statement II Number of parabolas with a given point as vertex and length of latusrectum equal to 4 is 2.

32 Consider the equation of the parabola is $y^2 = 4ax$.

Statement I Length of focal chord of a parabola having focus (2, 0) making an angle of 60° with X-axis is 32.

Statement II Length of focal chord of a parabola $y^2 = 4ax$ making an angle α with X-axis is $4a\operatorname{cosec}^2\alpha$.

33 Consider the equation of the parabola is $y^2 = 4ax$.

Statement I Area of triangle formed by pair of tangents drawn from a point (12, 8) to the parabola having focus (1,0) and their corresponding chord of contact is 32 sq units.

Statement II If from a point $P(x_1, y_1)$ tangents are drawn to a parabola, then area of triangle formed by these tangents and their corresponding chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{4\,|a|}$ sq units.

34 **Statement I** The latusrectum of a parabola is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$, then the equation of directrix of the parabola is $4x - 3y + 8 = 0$.

Statement II If P is any point on the parabola and PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then $(PM)^2 = (\text{latusrectum})(PN)$.

35 A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement I An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement II If the line, $y = mx + \frac{\sqrt{5}}{m}$ where, $m \neq 0$ is the common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

→ **JEE Mains 2013**

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 A ray of light moving parallel to the X-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point

(a) (0, 2) (b) (2, 0)
(c) (0, –2) (d) (–1, 2)

2 Mutually perpendicular tangents TA and TB are drawn to the parabola $y^2 = 8x$. The minimum length of AB is

(a) 16 (b) 4
(c) 8 (d) None of these

3 If a line $x + y = 1$ cuts the parabola $y^2 = 4x$ at points A and B and normals at A and B meet on C. The normals to the parabola from C, other than above two, meet the parabola in D, the coordinates of D are

(a) (2, 1) (b) (–4, 4)
(c) (4, 4) (d) None of these

4 A chord PP' of a parabola cuts the axis of the parabola at A. The feet of the perpendiculars from P and P' on the axis are M and M' respectively. If V is the vertex, then VM, VA, VM' are in

(a) AP (b) GP
(c) HP (d) None of these

5 The set of points on the axis of the parabola $y^2 = 4x + 8$ from which the 3 normals to the parabola are all real and different, is

(a) $\{(k, 0) | k \le -2\}$ (b) $\{(k, 0) | k > -2\}$
(c) $\{(k, 0) | k > 0\}$ (d) None of these

6 Normals drawn to $y^2 = 4ax$ at the points where it is intersected by the line $y = mx + c$, intersect at the point P. Foot of another normal drawn to the parabola from the point P may be

(a) $(a/m^2, -2a/m)$
(b) $(9a/m^2, -6a/m)$
(c) $(4a/m^2, -4a/m)$
(d) None of the above

7 Sides of an equilateral triangle ABC touch the parabola $y^2 = 4ax$, then points A, B, C lie on

(a) $y^2 = 3(x+a)^2 + 4ax$
(b) $y^2 = (x+a)^2 + ax$
(c) $y^2 = 3(x+a)^2 + ax$
(d) None of the above

8 Minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is

(a) $\sqrt{21}$ (b) $\sqrt{5}$
(c) $2\sqrt{7} - \sqrt{5}$ (d) None of these

9 The triangle formed by the tangent to the parabola $y = x^2$ at the point whose abscissa is x_0, $1 \le x_0 \le 2$, the Y-axis and the straight line $y = x_0^2$ has the greatest area if $x_0 =$

(a) 1 (b) 2
(c) 3/2 (d) None of these

10 The equation of the curve obtained by reflecting the parabola $y^2 = 4x$ about the line $x - y + 13 = 0$ is

(a) $(2y - x - 13)^2 = 4(y + 13)$
(b) $(2y + x - 13)^2 = 4(y - 13)$
(c) $(2y - x - 13)^2 = 4(y - 13)$
(d) None of the above

11 Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y+6)^2 = 1$, Then the equation of the circle, passing through C and having its centre at P is → **JEE Mains 2016**

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$

12 The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the line $y = |x|$ is → **JEE Mains 2017**

(a) $4(\sqrt{2} - 1)$ (b) $4(\sqrt{2} + 1)$
(c) $2(\sqrt{2} + 1)$ (d) $2(\sqrt{2} - 1)$

13 If y_1, y_2 are the ordinates of two points P and Q on the parabola and y_3 is the ordinate of the point of intersection of tangents at P and Q, then

(a) y_1, y_2, y_3 are in AP
(b) y_1, y_3, y_2 are in AP
(c) y_1, y_2, y_3 are in GP
(d) y_1, y_3, y_2 are in GP

14 The number of points with integral coordinates that lie in the interior of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 4x$ is

(a) 8 (b) 10
(c) 16 (d) None of these

15 The tangent and normal at the point $P = (16, 16)$ to the parabola $y^2 = 16x$ intersect the X-axis at the points Q and R respectively. The equation to the circum circle of ΔPQR is

(a) $x^2 + y^2 - 8x - 384 = 0$
(b) $x^2 + y^2 - 2x - 8y - 352 = 0$
(c) $x^2 + y^2 + 2y - 544 = 0$
(d) None of the above

ANSWERS

SESSION 1	**1** (b)	**2** (b)	**3** (a)	**4** (c)	**5** (c)	**6** (d)	**7** (a)	**8** (a)	**9** (c)	**10** (a)
	11 (c)	**12** (c)	**13** (d)	**14** (b)	**15** (b)	**16** (a)	**17** (d)	**18** (b)	**19** (d)	**20** (a)
	21 (d)	**22** (b)	**23** (a)	**24** (a)	**25** (c)	**26** (c)	**27** (a)	**28** (c)	**29** (a)	**30** (c)
	31 (c)	**32** (d)	**33** (c)	**34** (d)	**35** (b)					
SESSION 2	**1** (a)	**2** (c)	**3** (c)	**4** (b)	**5** (c)	**6** (c)	**7** (a)	**8** (b)	**9** (b)	**10** (d)
	11 (a)	**12** (a)	**13** (b)	**14** (d)	**15** (a)					

Hints and Explanations

SESSION 1

1 Axis is vertical i.e. parallel to Y-axis so its equation should be

$(x+1)^2 = 4a(y+2)$

It passes through (3, 6) so $4a = 2$.

Hence the equation of the required parabola is $x^2 + 2x - 2y - 3 = 0$,

2 Length of focal chord $= 4a \operatorname{cosec}^2\alpha$.

Here $a = 2$, $\alpha = \tan^{-1} 3$ i.e. $\tan\alpha = 3$.

$\therefore$ Length of focal chord

$= 4 \times 2 \times (1 + 1/9) = 80/9$.

3 Let vertex be (b, c). Then equation of parabola is $(x-b)^2 = 4a(y-c)$. It passes through the points (0, 4), (1, 9) and (−2, 6).

$\therefore \quad b^2 = 4a(4-c)$

$(1-b^2) = 4a(9-c)$

and $(-2-b)^2 = 4a(6-c)$.

Solving these equations, latus rectum $4a = 1/2$.

4 $y^2 - kx + 8 = 0 \Rightarrow y^2 = k(x - 8/k)$.

$\therefore$ Directrix is $x - 8/k = -k/4$

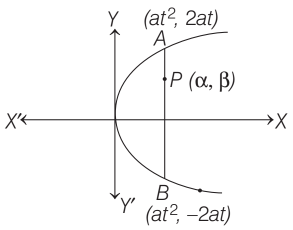
or $x = 8/k - k/4 = 1$

$\Rightarrow k^2 + 4k - 32 = 0 \Rightarrow k = -8$ or 4.

$\therefore$ One value of k is 4.

5 Let $P(\alpha, \beta)$ be the trisection point.

$\therefore \quad \alpha = at^2, \beta = \dfrac{2(2at) + 1(-2at)}{3}$



$\Rightarrow \quad \beta = \dfrac{2}{3}at \quad \Rightarrow \quad t = \dfrac{3\beta}{2a}$

$\therefore \quad \alpha = a\left(\dfrac{3\beta}{2a}\right)^2 \Rightarrow 9\beta^2 = 4a\alpha$

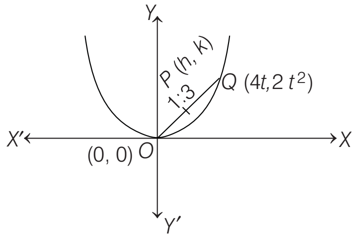
Hence, the locus of P is $9y^2 = 4ax$.

6 Any point on the parabola $x^2 = 8y$ is $(4t, 2t^2)$. Point P divides the line segment joining of $O(0,0)$ and $Q(4t, 2t^2)$ in the ratio 1: 3. Apply the section formula for internal division.

Equation of parabola is $x^2 = 8y$

Let any point Q on the parabola (i) is $(4t, 2t^2)$.

Let $P(h, k)$ be the point which divides the line segment joining (0,0) and $(4t, 2t^2)$ in the ratio 1:3.



$\therefore \quad h = \dfrac{1 \times 4t + 3 \times 0}{4} \Rightarrow h = t$

and $k = \dfrac{1 \times 2t^2 + 3 \times 0}{4} \Rightarrow k = \dfrac{t^2}{2}$

$\Rightarrow \quad k = \dfrac{1}{2}h^2 \qquad [\because t = h]$

$\Rightarrow \quad 2k = h^2 \Rightarrow 2y = x^2$, which is required locus.

7 Given, $(y-1)^2 = 4(x-1)$. P has coordinates $x = 1 + t^2$, $y = 1 + 2t$.

Tangent at P is

$(x-1) - (y-1)t + t^2 = 0$.

So, the directrix is $x = 0$.

$\therefore \quad Q = \left[0, t + 1 - \dfrac{1}{t}\right]$

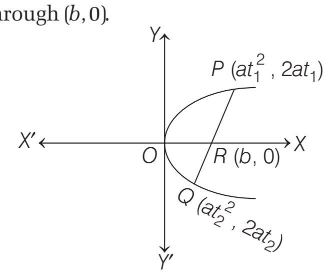
If $R(x, y)$ divides QP externally in the ratio 1 : 2.

$\therefore x = -(1 + t^2)$ and $y = 1 - \dfrac{2}{t} \Rightarrow t = \dfrac{2}{1-y}$

$\therefore \quad x + 1 + \dfrac{4}{(1-y)^2} = 0$

$\Rightarrow (x+1)(1-y)^2 + 4 = 0$

8 Line $x - b + \lambda y = 0$ always passes through $(b, 0)$.



Slope of PR = Slope of RQ

$\Rightarrow \quad t_1 t_2 = -\dfrac{b}{a}$

$\therefore$ Minimum value of $t_1 t_2 = -4$

and maximum value of $t_1 t_2 = -2$

9 The slope of the tangent to $y = x^2$ at (2, 4) is 4 and the equation of the tangent is $4x - y - 4 = 0$

Equation of the circle is

$(x-2)^2 + (y-4)^2 + \lambda(4x - y - 4) = 0 \ldots$(i)

Since, it passes through (0, 1).

Hence, $\lambda = \dfrac{13}{5}$

On putting the value of λ in Eq. (i), we get

$5(x-2)^2 + 5(y-4)^2 + 13(4x - y - 4) = 0$

$\Rightarrow 5(x^2 + 4 - 4x) + 5(y^2 + 16 - 8y)$

$+ 52x - 13y - 52 = 0$

$\Rightarrow 5x^2 + 5y^2 + 32x - 53y + 48 = 0$

$\Rightarrow x^2 + y^2 + \dfrac{32}{5}x - \dfrac{53}{5}y + \dfrac{48}{5}$

So, the centre of the circle is $\left(\dfrac{-16}{5}, \dfrac{53}{10}\right)$.

10 Tangent to parabola is $y = mx - am^2$.

$\therefore$ Tangents to two given parabolas are

$y = mx - (m^2/4)$ and

$y = m(x-2) + (m^2/4)$

These are identical $\Rightarrow m = 0$ or 4.

$\therefore$ Common tangents are $y = 0$ and $y = 4x - 4$.

11 $y = mx + 1/m$ passes through (1, 4).

$\Rightarrow m^2 - 4m + 1 = 0$.

$\therefore \tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$= \dfrac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$

$= \dfrac{\sqrt{16-4}}{1+1} = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$\Rightarrow \theta = \pi/3$.

12 Both lines pass through $(-a, b)$ which is a point on the directrix $x = -a$. Therefore, tangents drawn from $(-a, b)$ are perpendicular, so $m_1 m_2 = -1$.

13 The equation of any normal of $(x-2)^2 = 4(y-3)$ is

$(x-2) = m(y-3) - 2m - m^3$.

If it passes through $\left(1, \dfrac{h}{2}\right)$, then

$$1 - 2 = m\left(\frac{h}{2} - 3\right) - 2m - m^3$$

$\Rightarrow 2m^3 + m(10-h) - 2 = 0 = f(m)$ [say]

This equation will given three distinct values of m.

If $f'(m) = 0$ has two distinct roots, where

$f(m) = 2m^3 + m(10-h) - 2$

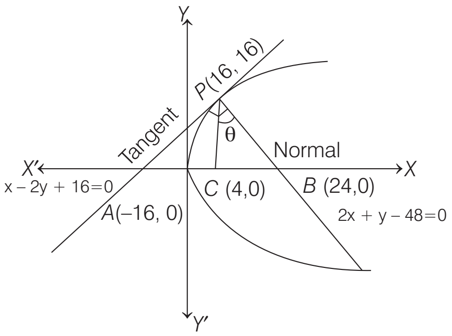
Now, $f'(m) = 6m^2 + (10-h)$

Put $f'(m) = 0 \Rightarrow m = \pm\sqrt{\dfrac{h-10}{6}}$

So, the values of m are real and distinct, if $h > 10$ i.e. $h \in (10, \infty)$.

14 Equation of tangent at $P(16, 16)$ is

$x - 2y + 16 = 0$



Slope of $PC = \dfrac{4}{3}$

Slope of $PB = -2$

Hence, $\tan\theta = \left|\dfrac{\frac{4}{3} + 2}{1 - \frac{4}{3} \times 2}\right| = 2$

15 Any point on the parabola is $P = (t^2, 2t)$.

Q is its image of the line $x - y + 1 = 0$.

$\therefore \dfrac{x - t^2}{1} = \dfrac{y - 2t}{-1} = -(t^2 - 2t + 1)$

$\Rightarrow Q = (2t - 1, t^2 + 1)$

Since, it lies on the line

$2x + y + 4 = 0$

$\therefore 4t - 2 + t^2 + 1 + 4 = 0$

$\Rightarrow t^2 + 4t + 3 = 0 \Rightarrow t = -1, -3$

So, the possible positions of P are $(1, -2)$ and $(9, -6)$.

16 $y = \dfrac{1}{3}a^3x^2 + \dfrac{a^2}{2}x - 2a$

$\Rightarrow \left(x + \dfrac{3}{4a}\right)^2 = \dfrac{3}{a^3}\left(y + \dfrac{35}{16}a\right)$

Vertex $P(h, k) = \left(-\dfrac{3}{4a}, -\dfrac{35}{16}a\right)$

$\Rightarrow a = -\dfrac{3}{4h}, a = -\dfrac{16k}{35}$

$\Rightarrow$ Locus of vertex P is $xy = 105/64$.

17 Let us recall that two parabolas are equal, if the length of their latusrectum are equal.

Length of the latusrectum of $y^2 = \lambda x$ is λ.

The equation of the second parabola is

$25\{(x-3)^2 + (y+2)^2\} = (3x - 4y - 2)^2$

$\Rightarrow \sqrt{(x-3)^2 + (y+2)^2} = \dfrac{|3x - 4y - 2|}{\sqrt{3^2 + 4^2}}$

Clearly, it represents a parabola having focus at $(3, -2)$ and equation of the directrix as $3x - 4y - 2 = 0$.

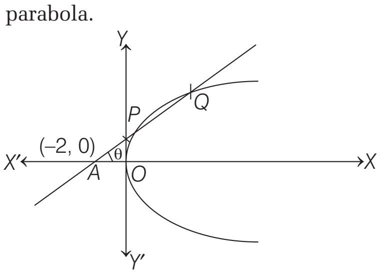
$\therefore$ Length of the latusrectum

$= 2$ (Distance between focus and directrix)

$= 2\left|\dfrac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}}\right| = 6$

Thus, the two parabolas are equal, if $\lambda = 6$.

18 Let $P(-2 + r\cos\theta, r\sin\theta)$ and P lies on parabola.



$\Rightarrow r^2\sin^2\theta - 4(-2 + r\cos\theta) = 0$

$\Rightarrow r_1 + r_2 = \dfrac{4\cos\theta}{\sin^2\theta} \Rightarrow r_1 r_2 = \dfrac{8}{\sin^2\theta}$

$\therefore \dfrac{r_1 + r_2}{r_1 r_2} = \dfrac{1}{AP} + \dfrac{1}{AQ}$

$\Rightarrow \cos\theta < \dfrac{1}{2} \Rightarrow \tan\theta > \sqrt{3}$

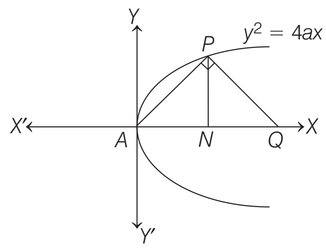
$\left[\text{because } \cos\theta \text{ is decreasing and } \tan\theta \text{ is increasing in } \left(0, \dfrac{\pi}{2}\right)\right]$

$\Rightarrow m > \sqrt{3}$

19 Let $P \equiv (at^2, 2at)$

Equation of the line PQ is

$y - 2at = -\dfrac{t}{2}(x - at^2)$.



On putting $y = 0$, we get $x = 4a + at^2$

So, the coordinates of Q and N are $(4a + at^2, 0)$ and $(at^2, 0)$, respectively.

So, length of projection

$= 4a + at^2 - at^2 = 4a$

20 Let the tangent to parabola be $y = mx + a/m$, if it touches the other curve, then $D = 0$, to get the value of m.

For parabola, $y^2 = 4x$

Let $y = mx + \dfrac{1}{m}$ be tangent line and it touches the parabola $x^2 = -32y$.

$\therefore x^2 = -32\left(mx + \dfrac{1}{m}\right)$

$\Rightarrow x^2 + 32mx + \dfrac{32}{m} = 0$

$\because D = 0$

$\therefore (32m)^2 - 4\left(\dfrac{32}{m}\right) = 0 \Rightarrow m^3 = \dfrac{1}{8}$

$\therefore m = \dfrac{1}{2}$

21 The tangents and normals form a rectangle.

Hence, tangents meet on the directrix.

Now, $(y-2)^2 = 2(x+2)$

Vertex $= (-2, 2)$ and directrix, $x = -\dfrac{5}{2}$

$\Rightarrow 2x + 5 = 0$

22 Here, $a = 2$ normal at t is

$xt + y = 2t^3 + 4t$. Focus $= (2, 0)$.

So, the point on the axis is (10, 0).

Normal passes through (10, 0).

$\therefore \quad 10 = 2t^2 + 4 \Rightarrow t^2 = 3$

So, the normal is at the point $(6, 4\sqrt{3})$.

So, the required length is

$\sqrt{(10-6)^2 + (4\sqrt{3})^2} = \sqrt{16+48} = 8$

23 The equation of normal to the parabola $y^2 = 12x$ with slope -1 is

$$y = -x - 2(3)(-1) - 3(-1)^3$$

$\Rightarrow \quad y = -x + 9 \Rightarrow x + y = 9$

$\therefore \quad k = 9$

Since, the focus of the parabola is (3, 0).

$\therefore \quad p = \left|\dfrac{3-9}{\sqrt{2}}\right|$

$\Rightarrow \quad 2p^2 = 36$

$\therefore \quad 3k^3 + 2p^2 = 3(9)^3 + 36 = 2223$

24 Solving $y^2 = 4ax$ and $x^2 = 4ay (a \neq 0)$, points of intersection are (0, 0) and $(4a, 4a)$.

Both points lie on the line

$$2bx + 3cy + 4d = 0$$

$\Rightarrow \quad d = 0$

and $\quad 2b + 3c = 0 \quad (\because a \neq 0)$

$\therefore \quad d^2 + (2b + 3c)^2 = 0$

25 .The normal $y = mx - 2am - am^3$ passes through $(4a, 0)$.

$\therefore \quad m^3 - 2m = 0 \Rightarrow m = 0, \pm\sqrt{2}$

$\therefore \quad$ Slopes of normals are $-\sqrt{2}, 0, \sqrt{2}$ which are in AP.

26 The coordinate of end of the latusrectum is $(a, 2a)$. The equation of the tangent at $(a, 2a)$ is $y \cdot 2a = 2a(x + a)$, i.e. $y = x + a$. The normal at $(a, 2a)$ is $y + x = 2a + a$, i.e. $y + x = 3a$.

On solving $y = 0$ and $y = x + a$, we get

$$x = -a, y = 0$$

On solving $y = 0$ and $y + x = 3a$, we get

$$x = 3a, y = 0$$

The area of the triangle with vertices $(a, 2a), (-a, 0), (3a, 0)$

$$= \frac{1}{2} \times 4a \times 2a = 4a^2$$

27 Equation of tangent to $y^2 = 8x$ at (2, 4) is

$4y = 4(x + 2)$ i.e. $x - y + 2 = 0 \quad \ldots$(i)

Let mid-point of QR be (x_1, y_1). Then, equation of $QR (T = S_1)$ is

$yy_1 - 4(x + x_1) - 5 = y_1^2 - 8x_1 - 5$

$\Rightarrow \quad 4x - yy_1 - 4x_1 + y_1^2 = 0 \quad \ldots$(ii)

On comparing Eqs. (i) and (ii), we get

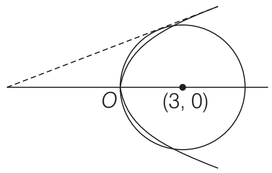
$$\frac{4}{1} = \frac{y_1}{1} = \frac{y_1^2 - 4x_1}{2}$$

$\Rightarrow \quad y_1 = 4$ and $8 = -4x_1 + y_1^2$

$\Rightarrow \quad y_1 = 4$ and $x_1 = 2$

Hence, required mid-point is (2, 4).

28 As common tangent is above X-axis, its slope is positive.



$y = mx + 1/m$ is a tangent to the parabola.

It touches the circle $(x-3)^2 + y^2 = 9$ if

$$\left|\frac{3m - 0 + 1/m}{\sqrt{1+m^2}}\right| = 3$$

$\Rightarrow \quad (3m + 1/m)^2 = 9(1 + m^2)$

$\Rightarrow \quad 6 + 1/m^2 = 9$ i.e. $m^2 = 1/3$.

As $m > 0$, $m = 1/\sqrt{3}$.

$\therefore$ Equation of common tangent above X-axis is

$$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$$

$\Rightarrow \quad \sqrt{3}y = x + 3.$

29 Let $P(x_1, y_1)$. Equation of any tangent making angle θ with X-axis (slope $= \tan\theta$) is

$y = x\tan\theta + \dfrac{1}{\tan\theta} \quad (\because y = mx + a/m)$

It passes through $P(x_1, y_1)$

$\therefore \quad x_1 \tan\theta + \dfrac{1}{\tan\theta} = y_1$

$\Rightarrow \quad x_1 \tan^2\theta - y_1 \tan\theta + 1 = 0$

$\tan\theta_1 + \tan\theta_2 = y_1/x_1$,

$\tan\theta_1 \tan\theta_2 = 1/x_1$

Given that, $\cot\theta_1 + \cot\theta_2 = 2$

$\Rightarrow \tan\theta_1 + \tan\theta_2 = 2\tan\theta_1 \tan\theta_2$

$\Rightarrow \quad y_1/x_1 = 2/x_1$

$\Rightarrow \quad$ Locus of $P(x_1, y_1)$ is $y = 2$.

30 Equation of chord of contact of (1,3) to the parabola $y^2 = 4x$ is

$3y = 2(x + 1) \quad \ldots$(i)

Solving Eq. (i) and parabola, we get

$$\frac{4}{9}(x+1)^2 = 4x$$

$\Rightarrow \quad x^2 - 7x + 1 = 0$

$\therefore \quad x_1 + x_2 = 7, x_1 x_2 = 1$

$\Rightarrow \quad (x_1 - x_2)^2 = 49 - 4 = 45$

Also, $\quad y_1 - y_2 = \dfrac{2}{3}(x_1 - x_2)$

$\Rightarrow \quad (y_1 - y_2)^2 = 20$

Hence, length of chord of contact

$$= \sqrt{45 + 20} = \sqrt{65}$$

31 Image of $(a, 0)$ with respect to tangent $yt = x + at^2$ is $(-a, 2at)$.

So, perpendicular bisector of $(a, 0)$ and $(-a, 2at)$ is the tangent line $yt = x + at^2$ to the parabola.

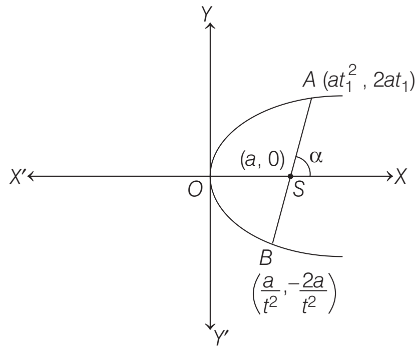
Hence, Statement I is true.

Statement II Infinitely many parabolas are possible.

Hence, Statement II is false.

32 Let AB be a focal chord slope of

$$AB = \frac{2t}{t^2 - 1} = \tan\alpha$$



$\Rightarrow \quad \tan\dfrac{\alpha}{2} = \dfrac{1}{t}$

$\Rightarrow \quad t = \cot\dfrac{\alpha}{2}$

Length of $AB = a\left(t + \dfrac{1}{t}\right)^2$

$$= 4a\,\text{cosec}^2\alpha$$

When $a = 2, \alpha = 60°$

$\therefore$ Length of $AB = 4(2)\,\text{cosec}^2(60°)$

$$= \frac{32}{3}$$

33 **Statement II** Area of triangle formed by these tangents and their corresponding chord of contact is $\dfrac{(y_1^2 - 4ax_1)^{3/2}}{2|a|}$.

Hence, Statement II is false.

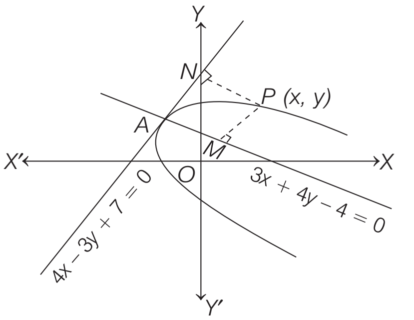
Statement I $x_1 = 12, y_1 = 8$

$\therefore \quad \text{Area} = \dfrac{(y_1^2 - 4ax_1)^{3/2}}{2}$

$$= \frac{(64-48)^{3/2}}{2} = 32$$

Hence, Statement I is true.

34 Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then



$(PM)^2 = (\text{Latusrectum})(PN)$

$\Rightarrow \left(\dfrac{3x + 4y - 4}{\sqrt{3^2 + 4^2}}\right)^2 = 4\left(\dfrac{4x + 3y + 7}{\sqrt{4^2 + (-3)^2}}\right)$

$\Rightarrow Y^2 = 4AX$

$\therefore \quad A = 1, Y = \dfrac{3x + 4y - 4}{5},$

$X = \dfrac{4x - 3y + 7}{5}$

So, the directrix is $X + A = 0$.

$\Rightarrow \dfrac{4x - 3y + 7}{5} + 1 = 0$

$\Rightarrow 4x - 3y + 12 = 0$

35 Equation of circle can be rewritten as

$x^2 + y^2 = \dfrac{5}{2}$

Let common tangent be $y = mx + \dfrac{\sqrt{5}}{m}$

So, the perpendicular from centre to the tangent is equal to radius.

$\therefore \quad \dfrac{\frac{\sqrt{5}}{m}}{\sqrt{1 + m^2}} = \sqrt{\dfrac{5}{2}}$

$\Rightarrow m\sqrt{1 + m^2} = \sqrt{2}$

$\Rightarrow m^2(1 + m^2) = 2$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow m = \pm 1$

$[\because m^2 + 2 \neq 0, \text{ as } m \in R]$

$\therefore \quad y = \pm x \pm \sqrt{5}$

Both statements are correct as

$m = \pm 1$

satisfies the given equation of Statement II.

But, Statement II is not a correct explanation of Statement I.

SESSION 2

1 Equation of axis is $y = 2$ which is parallel to X-axis.

Therefore, reflected ray will pass through the focus, which is (0, 2)

2 Tangents TA, TB are perpendicular

$\Rightarrow AB$ is focal chord

$\Rightarrow AB$ is latusrectum.

$\therefore \quad AB = 8$

3 Here A, B and D are co-normal points,

Let A, B and D be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. AB is the chord $x + y = 1$.

Solving $x + y = 1$ and $y^2 = 4x$, we get

$\Rightarrow \quad y^2 = 4(1 - y)$

$\Rightarrow \quad y^2 + 4y - 4 = 0$

$\therefore \quad y_1 + y_2 = -4$

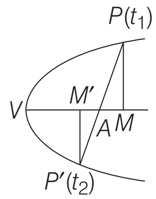
$y_1 + y_2 + y_3 = 0$

$\Rightarrow \quad y_3 = 4$

$\therefore \; 16 = 4x_3 \Rightarrow x_3 = 4.$

Hence, D is (4, 4).

4 $VM = at_1^2$, $VM' = at_2^2$ and $VA = k$, then



$$\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ k & 0 & 1 \end{vmatrix}$$

$[\because P, A, P'$ are collinear$]$

$\Rightarrow k + at_1t_2 = 0$

$\Rightarrow VM \cdot VM' = (at_1t_2)^2 = k^2 = VA^2$

$\Rightarrow VM$, VA and VM' are in GP.

5 Let $P(k, 0)$ be a point on the axis on the parabola $y^2 = 4(x + 2)$

Equation of normal at $(-2 + t^2, 2t)$ is

$t(x + 2) + y = 2t + t^3$

$\Rightarrow \quad y + tx = t^3.$

This passes through $(k, 0)$

$\therefore \; t^3 - kt = 0$ or $t = 0, t^2 = k$

For three real and distinct normals $k > 0$.

$\therefore$ Set of all such point

$= \{(k, 0) \mid k > 0\}.$

6 Let $y = mx + c$ intersect $y^2 = 4ax$ at $A(t_1)$ and $B(t_2)$. Then

$m = \dfrac{2}{t_1 + t_2}$

$\Rightarrow t_1 + t_2 = 2/m$

Normals at A and B meet at P. Let another normal from P meet the parabola at $C(t_3)$.

Then A, B and C are co-normal points.

$\therefore \quad t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad t_3 = -2/m$

$\therefore \quad C$ may be $\left(\dfrac{4a}{m^2}, -\dfrac{4a}{m}\right)$.

7 Let the sides of the triangle touch the parabola $y^2 = 4ax$ at t_1, t_2 and t_3.

Tangent at t_1, t_2, t_3 meets in

$A(at_1t_2, a(t_1 + t_2))$, $B(at_1t_3, a(t_1 + t_3))$ and $C(at_1t_3, a(t_2 + t_3))$.

Triangle ABC is equilateral.

$m_{AB} = \dfrac{a(t_3 - t_2)}{at_1(t_3 - t_2)} = \dfrac{1}{t_1}$, and

$m_{AC} = 1/t_2.$

$\therefore \quad \sqrt{3} = \left|\dfrac{1/t_1 - 1/t_2}{1 + 1/t_1t_2}\right| = \dfrac{|t_2 - t_1|}{|1 + t_1t_2|}$

$\Rightarrow (t_2 - t_1)^2 = 3(1 + t_1t_2)^2$

$\Rightarrow (t_1 + t_2)^2 - 4t_1t_2 = 3 + 6t_1t_2 + 3(t_1t_2)^2$

Let A be (x, y). Then

$\dfrac{y^2}{a^2} = 3 + 10\dfrac{x}{a} + \dfrac{3x^2}{a^2}$

$\Rightarrow y^2 = 3a^2 + 10ax + 3x^2$

$= 3(x + a)^2 + 4ax$

8 Circle is $x^2 + y^2 - 12x + 31 = 0$, $C(6, 0)$, $r = \sqrt{5}$.

Shortest distance will take place along the common normal. Normal to $y^2 = 4x$ at $A(t^2, 2t)$ is

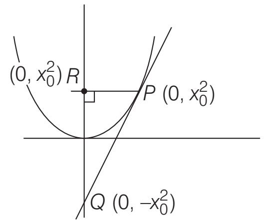
$y = -tx + 2t + t^3$. It must passes through (6, 0).

$\therefore \quad t^3 - 4t = 0 \Rightarrow t = 0$ or $\pm\sqrt{2}$.

$\therefore$ Distances between the curves along common normal are $6 - \sqrt{5}$, and $\sqrt{5}$.

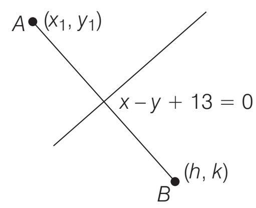
Hence, minimum distance between the curves $= \sqrt{5}$.

9 Area $A = 2x_0^2 \times x_0 \times \dfrac{1}{2} = x_0^3$



Since $1 \le x_0 \le 2$, then area is max. at $x_0 = 2$.

10 Let $A(x_1, y_1)$ be any point on the parabola



$y^2 = 4x$ and $B(h, k)$ be the reflection of A with respect to the line $x - y + 13 = 0$. Then,

$\dfrac{h + x_1}{2} - \dfrac{k + y_1}{2} + 13 = 0$

and $\dfrac{k - y_1}{h - x_1} \cdot (1) = -1$

Then, $x_1 = k - 13$, $y_1 = h + 13$

$\therefore \quad (h+13)^2 = 4(k-13)$

$\therefore$ Locus of the point B is

$$(x+13)^2 = 4(y-13).$$

11 Normal at $P(at^2, 2at)$ is

$y + tx = 2at + at^3$

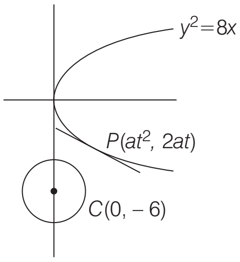
Given it passes $(0, -6)$

$\Rightarrow \quad -6 = 2at + at^3 \quad [\because a = 2]$

$\Rightarrow \quad -6 = 4t + 2t^3$

$\Rightarrow \quad t^3 + 2t + 3 = 0$

$\Rightarrow \quad t = -1$



So, $P(a, -2a) = (2, -4) \quad [\because a = 1]$

Radius of circle

$$= CP = \sqrt{2^2 + (-4+6)^2} = 2\sqrt{2}$$

Equation of circle is

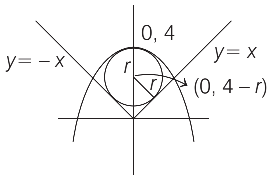
$$(x-2)^2 + (y+4)^2 = (2\sqrt{2})^2$$

$$x^2 + y^2 - 4x + 8y + 12 = 0$$

12 $x^2 + x - 4 = 0$

$$x = \frac{-1 \pm \sqrt{1+16}}{2}$$

$$x = \frac{-1+\sqrt{17}}{2}$$



$$\frac{4 - r - 0}{\sqrt{2}} = r$$

$$4 - r = \pm\sqrt{2}r$$

$$r = \frac{4}{\sqrt{2}+1} \quad \left(\because \frac{4}{1-\sqrt{2}} < 0\right)$$

$$r = 4(\sqrt{2} - 1)$$

13 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Tangents at P and Q to the parabola $y^2 = 4ax$ are

$yy_1 = 2a(x + x_1)$

and $yy_2 = 2a(x + x_2)$.

$\therefore \quad y(y_1 - y_2) = 2a(x_1 - x_2)$

$$= \frac{2a(y_1^2 - y_2^2)}{4a}$$

$$\Rightarrow \quad y_3 = \frac{y_1 + y_2}{2}$$

$\Rightarrow \quad y_1, y_3, y_2$ are in AP.

14 Let (p, q), $p, q \in Z$ be an interior point of both the curves.

Then, $\quad p^2 + q^2 - 16 < 0$

and $q^2 - 4p < 0$, $p \geq 0$.

$\Rightarrow \quad p > (q/2)^2$ and $p^2 < 16 - q^2$.

$q = 0 \Rightarrow p = 1, 2, 3$

$q = 1 \Rightarrow p = 1, 2, 3$

$q = 2 \Rightarrow p = 2, 3$

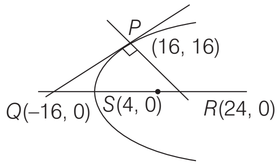
$q = 3 \Rightarrow p =$ has no value.

$\therefore$ There only 8 points (1, 0), (2, 0), (3, 0), (1, 1), (2, 1), (2, 2) (3, 1), (3, 2) in upper half.

Due to symmetry about X-axis.

(1, – 1), (2, – 1), (2, – 2), (3, – 1), (3, – 2) are also interior points. Hence in all, three are 13 interior integral points.

15 Clearly, QR is the diameter of the required circle.



$16y = 8(x + 16) \Rightarrow Q = (-16, 0)$

$y - 16 = -2(x - 16) \Rightarrow R = (24, 0)$

$\therefore$ Equation of required circle is

$(x + 16)(x - 24) + y^2 = 0$

$\Rightarrow \quad x^2 + y^2 - 8x - 384 = 0$